RESEARCH NOTES

On a Result of Birnbaum regarding the Skewness of X in a Bivariate Normal Population

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In a recent paper[1] entitled "Effect of Linear Truncation on a Multinormal Population," Z. W. Birnbaum obtained the following expression for the third moment of X, truncating the distribution in Y to the set $Y \geqslant x > 0$ as

$$E[X - E(X)]^3 = E(X) . \rho^2 \{[\lambda(x) - x][2\lambda(x) - x] - 1\}. \quad (1)$$

where

$$\frac{1}{\lambda(x)} = e^{\frac{x^2}{2}} \int_x^{\infty} e^{-t^2/2}\, dt \quad (2)$$

and then, based on computational evidence, conjectured that the quantity in braces is always > 0, and stated that "no analytic proof of the above statement has been obtained".

The purpose of the present note is to supply an analytic proof of the above statement and thereby confirm the conjecture that the Skewness of X, after truncation has the same sign as $E(X)$.

Inequalities for expressions involving $\lambda(x)$ besides the one considered above have received attention in Statistical literature. Z. W. Birnbaum[2] and R. D. Gordan[3] obtained the following inequalities for

$$\frac{1}{\lambda(x)} = R(x) \text{ say} \quad (3)$$

as follows:

$$x/x^2 + 1 \leqslant R_x \leqslant \frac{1}{x}, \text{ for } x > 0 \text{ (R. D. Gordon)}$$

$$\frac{\sqrt{4 + x^2} - x}{2} \leqslant R_x, \text{ for } x > 0 \text{ (Z. W. Birnbaum).} \quad (5)$$

Birnbaum's result is an improvement of Gordan's as far as the lower inequality is considered. The result of the Lemma of this paper is more powerful than the results of Gordan and Birnbaum. Incidentally it is shown that inequalities for $R(x)$ for any desired degree of approximation can be obtained as in the Lemma of the present paper.

Lemma.—Laplace's[4] continued fraction for the Normal probability integral

$$I_x = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \tag{6}$$

gives

$$I_x = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \cdots \right\} \tag{7}$$

Hence

$$R_x = \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \cdots \right\}, \tag{8}$$

where

$$R_x = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt \tag{9}$$

In a continued fraction the convergents of an odd order are all greater than the continued fraction, but continually decrease, and the convergents of an even order are all less than the continued fraction but continually increase.

The first four convergents of the continued fraction (8) are respectively equal to

$$\frac{1}{x}, \; \frac{x}{x^2+1}, \; \frac{x^2+2}{x^3+3x}, \text{ and } \frac{x^3+5x}{x^3+6x^2+3} \tag{10}$$

Hence we have

$$x/x^2 + 1 \leqslant R_x \leqslant \frac{1}{x} \tag{11}$$

and

$$\frac{x^3+5x}{x^3+6x^2+3} \leqslant R_x \leqslant \frac{x^2+2}{x^3+3x}. \tag{12}$$

It is easily seen that,

$$(x^3+5x)/x^3 + 6x^2 + 3 \geqslant \frac{\sqrt{4+x^2} - x}{2} \tag{13}$$

for sufficiently large values of $x > 0$. Thus we can get inequalities for R_x upto the desired degree of accuracy by calculating sufficient number of convergents of the continued fraction (8).

Proof of the Main Result.—The inequality

$$[\lambda(x) - x]\,[2\lambda(x) - x] - 1 > 0 \tag{14}$$

will be established if we show that

$$2 - 3x\,R(x) + (x^2 - 1)\,R^2(x) > 0 \tag{15}$$

From (11) and (12) we have

$$2 - 3x\,R(x) + (x^2 - 1)\,R^2(x) > 2 - 3x \cdot \frac{x^2 + 2}{x^3 + 3x}$$
$$+ (x^2 - 1) \cdot \frac{x}{x^2 + 1} \cdot \frac{x^3 + 5x}{x^3 + 6x^2 + 3}$$

which is easily seen to be > 0 for sufficiently large $x > 0$. Hence

$$[\lambda(x) - x]\,[2\lambda(x) - x] - 1 > 0$$

which shows that the third moment of X, about its mean which is a measure of Skewness has the same sign as $E(X)$.

References

1. Birnbaum, Z. W. .. *Annals of Math. Statistics*, 1950, **21**, 272–79.
2. ———— .. *Ibid.*, 1942, **13**, 245–46.
3. Gordan, R. D. .. *Ibid.*, 1941, **12**, 364–66.
4. Kendall .. *Advanced Theory of Statistics*, **1**, 129–30.